Abstract carrier space formalism for the irreducible tensor operators of compact quantum group algebras

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(October 12, 1995)

Abstract

Defining conditions for irreducible tensor operators associated with the unitary irreducible corepresentations of compact quantum group algebras are deduced within the framework of the abstract carrier space formalism. It is shown that there are *two* types of irreducible tensor operator, which may be called 'ordinary' and 'twisted'. The consistency of the definitions is demonstrated, and various consequences are deduced, including generalizations of the Wigner-Eckart theorem for both the ordinary and twisted operators. Examples of irreducible tensor operators for the standard deformation of the function algebra of the compact Lie group $SU(2)$ are described to demonstrate the applicability of the new definitions.

Typeset using REVTEX

Running title: Irreducible tensor operators

I. INTRODUCTION

Most of the applications to physics of the theories of groups and Lie algebras depend on the Wigner-Eckart theorem, and so it is of great interest to see how this theorem generalizes to other algebraic structures. In a previous paper[1], hereafter referred to as Paper I, a study of the definitions and properties of irreducible tensor operators for a *compact quantum group algebra* $\mathcal{A}$ was initiated by examining the case of the right regular and left regular coaction formalisms, and was extended to the case of operators associated with the corresponding quantum homogeneous spaces of $\mathcal{A}$. In the present paper this will be further extended to the case of operators acting in the abstract carrier spaces of irreducible corepresentations of $\mathcal{A}$.

The plan of the present paper is as follows. The remainder of this section will be devoted to putting the analysis that follows into context, first by reviewing briefly the situation for compact Lie groups, and then by indicating the background for the generalization of compact Lie groups to compact quantum group algebras. In the next section the most relevant properties of compact quantum group algebras will be summarized, particular attention being devoted to the essential role played by corepresentations. This summary is continued in Section III, where the two different types of tensor product of corepresentations and their associated Clebsch-Gordan coefficients are briefly discussed. The heart of the paper is reached in Section IV, where the irreducible tensor operators are defined and some of their immediate properties are deduced. In particular, it will be shown there that there are *two* types of irreducible tensor operators, which will be described as being *ordinary* and *twisted* respectively. The *motivations* for the definitions of Section IV are deliberately relegated to the Appendix in order to emphasize that the treatment given for the compact quantum group algebras in Sections IV and V entirely self-contained. In Section V it is shown that there are *two* theorems of the Wigner-Eckart type, one for the 'ordinary' and one for the 'twisted' irreducible tensor operators. To illustrate the applicability of the new definitions of Section IV, examples of irreducible tensor operators for the standard deformation of

the function algebra of the compact Lie group $SU(2)$ are described in Section VI. Unless otherwise stated, the notations, definitions, and terminology are exactly the same as those given in Paper I, which also contains an account of the relationship of the present line of study to previous work on the applicability of the Wigner-Eckart theorem to quantum groups.

Because the space of functions defined on a compact Lie group $\mathcal{G}$ is a special example of a compact quantum group algebra, all the well-known results for compact Lie groups naturally reappear in this particular case. However, as the detailed analysis shows, the theory in the general situation is rather more subtle, and exhibits various complications. Nevertheless, the point of view of the present communication is best introduced by considering first the abstract carrier space formalism in this very well established and familiar context of a compact Lie group $\mathcal{G}$ (c.f. Refs. 2, 3). Let V^p be a carrier space for a unitary irreducible representation $\mathbf{\Gamma}^p$ of $\mathcal{G}$, and let $\psi_1^p, \psi_2^p, \ldots, \psi_{d_p}^p$ be an ortho-normal basis for V^p. Define for each $T \in \mathcal{G}$ a linear operator $\Phi^p(T)$ that acts on V^p by the requirement that

$$\Phi^p(T)(\psi_n^p) = \sum_{m=1}^{d_p} \Gamma^p(T)_{mn} \psi_m^p \tag{1}$$

for all $T \in \mathcal{G}$ and all $n = 1, 2, \ldots, d_p$. Let $\mathbf{\Gamma}^p$, $\mathbf{\Gamma}^q$, and $\mathbf{\Gamma}^r$ be any three unitary irreducible resentations of $\mathcal{G}$. Then one can consider a set of irreducible tensor operators $Q_1^q, Q_2^q, \ldots, Q_{d_q}^q$ that each map V^p into V^r and which are such that

$$\Phi^r(T)\, Q_n^q\, \Phi^p(T)^{-1} = \sum_{m=1}^{d_q} \Gamma^q(T)_{mn} Q_m^q \tag{2}$$

for all $T \in \mathcal{G}$ and all $n = 1, 2, \ldots, d_q$. In this case the Wigner-Eckart theorem deals with inner products $\langle\ ,\ \rangle$ defined on V^r and states that the j, k, and ℓ dependence of $\langle \psi_\ell^r, Q_k^q(\psi_j^p)\rangle$ depends only on Clebsch-Gordan coefficients for the reduction of the tensor product $\mathbf{\Gamma}^p \otimes \mathbf{\Gamma}^q$ into its irreducible constituents $\mathbf{\Gamma}^r$.

In a minor extension of this formalism, one could introduce an inner product space V that is a direct sum of carrier spaces of certain unitary irreducible representations of $\mathcal{G}$ and which contains at least $V^p \oplus V^r$ (and which, in the extreme case, may contain one carrier

space for every inequivalent irreducible representation of $\mathcal{G}$). Then, for each $T \in \mathcal{G}$ an operator $\Phi(T)$ can be defined which maps elements of V into V, and which acts as $\Phi^p(T)$ on V^p, as $\Phi^r(T)$ on V^r, and so on. The irreducible tensor operators are then required to each map V into V and to be such that $\Phi(T)\, Q^q_n\, \Phi(T)^{-1} = \sum_{m=1}^{d_q} \Gamma^q(T)_{mn} Q^q_m$ for all $T \in \mathcal{G}$ and all $n = 1, 2, \ldots, d_q$. In this case the Wigner-Eckart theorem deals with inner products $\langle\ ,\ \rangle$ defined on V, but is otherwise the same as above.

As emphasized in Paper I, one most important lesson that can be drawn from these simple group theoretical considerations concerns the *consistency* of the definitions of the basis vectors *and* of the irreducible tensor operators. As $\Phi^p(T)\Phi^p(T') = \Phi^p(TT')$ and $\mathbf{\Gamma}^p(T)\mathbf{\Gamma}^p(T') = \mathbf{\Gamma}^p(TT')$ for all $T, T' \in \mathcal{G}$, it follows that if (1) is valid for T and for T', then it is also valid for their product TT'. Similarly, and very significantly, by defining for each $T \in \mathcal{G}$ an operator $\Psi(T)$ by

$$\Psi(T)(Q) = \Phi^r(T)\, Q\, \Phi^p(T)^{-1} \tag{3}$$

for every operator Q that maps V^p into V^r, the definition (2) can be recast as

$$\Psi(T)(Q^q_n) = \sum_{m=1}^{d_q} \Gamma^q(T)_{mn} Q^q_m \tag{4}$$

for all $T \in \mathcal{G}$ and all $n = 1, 2, \ldots, d_q$. As $\Psi(T)\Psi(T') = \Psi(TT')$ for all $T, T' \in \mathcal{G}$, it follows that if (4) is valid for T and for T', then it is also valid for their product TT'. Put another way, because of the similarity in form between (1) and (4), the *consistency* of the definition (2) of the irreducible tensor operators Q^q_n is ensured by the fact that they too form a basis for a carrier space, this time for $\mathbf{\Gamma}^q$. In the analysis that follows (cf. Section IV), essentially this argument will be used to justify the definitions that will be given for the irreducible tensor operators of the compact quantum group algebras, the only essential difference being that the argument has to be cast in terms of *co*representations instead of representations.

As is well known, the set of functions defined on a Lie group $\mathcal{G}$ form a Hopf algebra, $\mathcal{A}$, and the dual $\mathcal{A}'$ of $\mathcal{A}$ is the universal enveloping algebra of the Lie algebra $\mathcal{L}$ of $\mathcal{G}$. Moreover, the structure of $\mathcal{G}$ can be encoded into the structure of $\mathcal{A}$, and, in particular, $\mathcal{A}$ is

commutative. A 'deformation' (or 'quantization') of $\mathcal{A}'$ induces a corresponding deformation of $\mathcal{A}$, and will make $\mathcal{A}$ non- commutative as well as being non-cocommutative. Although most attention has been focused on the deformed Hopf algebras $\mathcal{A}'$, it has been demonstrated by the pioneering work of Woronowicz[4−6], which itself has been refined and developed by Dijkhuizen and Koornwinder[7−12], that it is of very great interest to produce a self-contained and direct study of generalizations of the Hopf algebras $\mathcal{A}$. This can be done by assuming that they have certain characteristic properties, and the resulting structures have been called *compact matrix pseudogroups* by Woronowicz[4−6], and *compact quantum group algebras* by Dijkhuizen and Koornwinder[7−12]. It is these that provide the framework for the present paper, which, as intimated above, is devoted to the study of the irreducible tensor operators for compact quantum group algebras in the abstract carrier space formalism.

II. COREPRESENTATIONS OF COMPACT QUANTUM GROUP ALGEBRAS

It should be recalled (c.f. Refs. 1, 7-12) that a *right* $\mathcal{A}$- *comodule* consists of a vector space V and a linear mapping π_V from V to $V \otimes \mathcal{A}$ such that

$$(\pi_V \otimes id) \circ \pi_V = (id \otimes \Delta) \circ \pi_V \tag{5}$$

and

$$((id \otimes \epsilon) \circ \pi_V)(v) = v \otimes 1_{\boldsymbol{C}} \tag{6}$$

for all $v \in V$. The operation π_V is then said to be a *right coaction* and provides a *corepresentation* of $\mathcal{A}$ with carrier space V.

Of great importance are the finite-dimensional *irreducible* corepresentations, which, for a compact quantum group algebra $\mathcal{A}$, are assumed to form a *countable* set (up to equivalence). Moreover each such irreducible corepresentation is equivalent to a *unitary* corepresentation. Let π^p, for $p = 1, 2, \ldots$, denote the set of unitary irreducible corepresentations of $\mathcal{A}$ (one being chosen from every equivalence class), and let V^p be a carrier space of π^p, assumed to

be of finite dimension d_p, with basis $v_1^p, v_2^p, \ldots, v_{d_p}^p$. Then there exists a uniquely determined set of elements π_{jk}^p of $\mathcal{A}$ (for $j, k = 1, 2, \ldots, d_p$), called the *matrix coefficients* of π^p, which are such that

$$\pi^p(v_j^p) = \sum_{k=1}^{d_p} v_k^p \otimes \pi_{kj}^p \tag{7}$$

for all $j = 1, 2, \ldots, d_p$. The requirements (5) and (6) then imply that

$$\Delta(\pi_{jk}^p) = \sum_{\ell=1}^{d_p} \pi_{j\ell}^p \otimes \pi_{\ell k}^p \tag{8}$$

and

$$\epsilon(\pi_{jk}^p) = \delta_{jk} \tag{9}$$

(for $j, k = 1, 2, \ldots, d_p$). The unitary requirement on π^p implies that

$$S(\pi_{jk}^p) = \pi_{kj}^{p*}, \tag{10}$$

$$\sum_{\ell=1}^{d_p} M(\pi_{\ell j}^{p*} \otimes \pi_{\ell k}^p) = \delta_{jk} 1_{\mathcal{A}}, \tag{11}$$

and

$$\sum_{\ell=1}^{d_p} M(\pi_{j\ell}^p \otimes \pi_{k\ell}^{p*}) = \delta_{jk} 1_{\mathcal{A}} \tag{12}$$

(for all $j, k = 1, 2, \ldots, d_p$). For a compact quantum group algebra $\mathcal{A}$ this set of matrix coefficients are assumed (c.f.Refs. 1, 7-12) to form a basis for $\mathcal{A}$.

Let $\mathcal{P}_{ij}^p$ be a set of projection operators for V^p that are defined by

$$\mathcal{P}_{ij}^p(v_k^p) = \delta_{ik}\, v_j^p \tag{13}$$

for all $i, j, k = 1, 2, \ldots, d_p$. Let π^r be another unitary irreducible corepresentation of $\mathcal{A}$, and let $\mathcal{L}^{pr}$ be the set of linear operators that map elements of V^p into V^r. A basis for $\mathcal{L}^{pr}$ is provided by the set of operators $\mathcal{P}_{ij}^{pr}$ that are defined by

$$\mathcal{P}_{ij}^{pr}(v_k^p) = \delta_{ik}\, v_j^r \tag{14}$$

for all $i, k = 1, 2, \ldots, d_p$ and all $j = 1, 2, \ldots, d_r$. Then

$$\mathcal{P}^r_{mn} \circ \mathcal{P}^{pr}_{k\ell} \circ \mathcal{P}^p_{ij} = \delta_{kj}\, \delta_{m\ell}\, \mathcal{P}^{pr}_{in} \tag{15}$$

for all $i, j, k = 1, 2, \ldots, d_p$ and all $\ell, m, n = 1, 2, \ldots, d_r$. If Q is any element of $\mathcal{L}^{pr}$, then

$$Q(v^p_k) = \sum_{j=1}^{d_r} q_{jk}\, v^r_j \tag{16}$$

for all $k = 1, 2, \ldots, d_p$, where q_{jk} are the complex numbers that are defined by

$$q_{jk} = \langle v^r_j, Q(v^p_k)\rangle \tag{17}$$

for all $k = 1, 2, \ldots, d_p$ and all $j = 1, 2, \ldots, d_r$, $\langle\ ,\ \rangle$ being the inner product of V^r. Moreover one can write

$$Q = \sum_{i=1}^{d_p} \sum_{j=1}^{d_r}\, q_{ji}\, \mathcal{P}^{pr}_{ij}\ . \tag{18}$$

III. TENSOR PRODUCTS AND CLEBSCH-GORDAN COEFFICIENTS

A. Ordinary and twisted tensor products

With the *ordinary tensor product* of two irreducible corepresentations π^p and π^q of $\mathcal{A}$ (with carrier spaces V^p and V^q respectively) being defined as the mapping $\pi^p \boxtimes \pi^q$ from $V^p \otimes V^q$ to $V^p \otimes V^q \otimes \mathcal{A}$ that is such that

$$\pi^p \boxtimes \pi^q = (id \otimes id \otimes M) \circ (id \otimes \sigma \otimes id) \circ (\pi^p \otimes \pi^q)\ , \tag{19}$$

it is easily shown from (7) that the corresponding matrix coefficients are given by

$$(\pi^p \boxtimes \pi^q)_{st,jk} = M(\pi^p_{sj} \otimes \pi^q_{tk}) \tag{20}$$

for all $j, s = 1, 2, \ldots, d_p$ and all $k, t = 1, 2, \ldots, d_q$.

Similarly, with the *twisted tensor product* of π^p and π^q being defined as the mapping $\pi^p \widetilde{\boxtimes} \pi^q$ from $V^p \otimes V^q$ to $V^p \otimes V^q \otimes \mathcal{A}$ that is such that

$$\pi^p \widetilde{\boxtimes} \pi^q = (id \otimes id \otimes M) \circ (id \otimes id \otimes \sigma) \circ (id \otimes \sigma \otimes id) \circ (\pi^p \otimes \pi^q) \ , \tag{21}$$

it is also easily shown from (7) that the corresponding matrix coefficients are given by

$$(\pi^p \widetilde{\boxtimes} \pi^q)_{st,jk} = M(\pi^q_{tk} \otimes \pi^p_{sj}) \tag{22}$$

for all $j, s = 1, 2, \ldots, d_p$ and all $k, t = 1, 2, \ldots, d_q$.

B. Clebsch-Gordan coefficients

Suppose that the ordinary tensor product $\pi^p \boxtimes \pi^q$ is reducible (and hence is completely reducible (c.f. Refs. 1, 7-12)), and that n^r_{pq} is the number of times that the irreducible corepresentation π^r (or a corepresentation equivalent to it) appears in its reduction. If the carrier spaces V^p and V^q have basis elements $v^p_1, v^p_1, \ldots, v^p_{d_p}$ and $v^q_1, v^q_1, \ldots, v^q_{d_q}$ respectively, then the set of elements $v^p_j \otimes v^q_k$ form a basis for $V^p \otimes V^q$, the carrier space of $\pi^p \boxtimes \pi^q$, and consequently appropriate linear combinations form bases for all the irreducible corepresentations π^r that appear in the reduction of the tensor product. Let $w^{r,\alpha}_\ell$ be such a combination, so that

$$w^{r,\alpha}_\ell = \sum_{j=1}^{d_p} \sum_{k=1}^{d_q} \left(\begin{array}{cc|cc} p & q & r \ , & \alpha \\ j & k & \ell & \end{array} \right) v^p_j \otimes v^q_k, \tag{23}$$

for $\ell = 1, 2, \ldots, d_r$, and $\alpha = 1, 2, \ldots, n^r_{pq}$, and

$$(\pi^p \boxtimes \pi^q)(w^{r,\alpha}_\ell) = \sum_{u=1}^{d_r} w^{r,\alpha}_u \otimes \pi^r_{ul}, \tag{24}$$

for $u = 1, 2, \ldots, d_r$, and $\alpha = 1, 2, \ldots, n^r_{pq}$. The inverse of (23) is

$$v^p_j \otimes v^q_k = \sum_r \sum_{\alpha=1}^{n^r_{pq}} \sum_{\ell=1}^{d_r} \left(\begin{array}{cc|cc} r \ , & \alpha & p & q \\ \ell & & j & k \end{array} \right) w^{r,\alpha}_\ell, \tag{25}$$

for $j = 1, 2, \ldots, d_p$ and $k = 1, 2, \ldots, d_q$. The *Clebsch-Gordan coefficients* defined in (23) form the elements of a $d_p \times d_q$ matrix $\mathbf{C}$, while the inverse coefficients defined in (25) form the elements of $\mathbf{C}^{-1}$, where

$$\mathbf{C}^{-1}(\pi^p \boxtimes \pi^q)\mathbf{C} = \sum_r \oplus n^r_{pq} \pi^r. \tag{26}$$

This implies that

$$(\pi^p \boxtimes \pi^q)_{is,jt} = \sum_r \sum_{\alpha=1}^{n^r_{pq}} \sum_{\ell,u=1}^{d_r} \left(\begin{array}{cc|cc} p & q & r & , \alpha \\ i & s & u & \end{array} \right) \pi^r_{u\ell} \left(\begin{array}{cc|cc} r & , \alpha & p & q \\ \ell & & j & t \end{array} \right) \tag{27}$$

for $i, j = 1, 2, \ldots, d_p$, and $s, t = 1, 2, \ldots, d_q$.

Thus, by (20), the product of any two basis elements of $\mathcal{A}$ can be expressed in terms of Clebsch-Gordan coefficients, for

$$M(\pi^p_{ij} \otimes \pi^q_{st}) = \sum_r \sum_{\alpha=1}^{n^r_{pq}} \sum_{\ell,u=1}^{d_r} \left(\begin{array}{cc|cc} p & q & r & , \alpha \\ i & s & u & \end{array} \right) \pi^r_{u\ell} \left(\begin{array}{cc|cc} r & , \alpha & p & q \\ \ell & & j & t \end{array} \right) \tag{28}$$

for $i, j = 1, 2, \ldots, d_p$, and $s, t = 1, 2, \ldots, d_q$. This is essentially the converse of the relation (I.137), that is, of

$$\begin{aligned} h(\pi^{r*}_{ul} \pi^q_{tk} \pi^p_{sj}) \; &= \; \textstyle\sum_{\alpha=1}^{n^r_{qp}} \sum_{v=1}^{d_r} \left(\begin{array}{cc|cc} r & , \alpha & q & p \\ \ell & & k & j \end{array} \right) \left(\begin{array}{cc|cc} q & p & r & , \alpha \\ t & s & v & \end{array} \right) \\ &\times \{ ((\mathbf{F}^r)^{-1})_{vu} / tr((\mathbf{F}^r)^{-1}) \} \end{aligned} \tag{29}$$

for all $j = 1, 2, \ldots, d_p$, $k = 1, 2, \ldots, d_q$, and $l = 1, 2, \ldots, d_r$. Here h is the Haar functional and $\mathbf{F}^r$ is a non-singular $d_r \times d_r$ matrix with the property that

$$\sum_{k=1}^{d_r} F^r_{jk} \pi^r_{k\ell} = \sum_{k=1}^{d_r} \pi^{r\ddagger}_{jk} F^r_{k\ell} \tag{30}$$

(for all $j, \ell = 1, 2, \ldots, d_r$), where $\pi^{r\ddagger}$ is the doubly contragredient partner of π^r.

IV. THE IRREDUCIBLE TENSOR OPERATORS

A. Introduction

Let π^p,π^q, and π^r be unitary irreducible right coactions of $\mathcal{A}$ of dimensions d_p, d_q, and d_r respectively, and with matrix coefficients π^p_{jk}, π^q_{jk}, and π^r_{jk} respectively. Let $\mathcal{L}^{pr}$ be the vector space of operators introduced in Section II. It will be shown that there exist two

types of irreducible tensor operators that are members of $\mathcal{L}^{pr}$ and which belong to the corepresentation π^q. These will be denoted by Q_j^q and $\widetilde{Q}_j^q$ (for $j = 1, 2, \ldots, d_q$), and will be called *ordinary* and *twisted* irreducible tensor operators respectively. Naturally the two types of irreducible tensor operators coincide in the special case in which $\mathcal{A}$ is commutative.

It will also be shown that the definitions of both of these types of irreducible tensor operators is easily extended to the case in which V is a vector space that is a direct sum of carrier spaces of unitary irreducible corepresentations of $\mathcal{A}$ and which contains at least $V^p \oplus V^r$.

B. Definitions of irreducible tensor operators

1. Definition of the ordinary irreducible tensor operators Q_j^q

The *ordinary irreducible tensor operators* Q_j^q belonging to the unitary irreducible right coaction π^q of $\mathcal{A}$ are *defined* to be members of $\mathcal{L}^{pr}$ that satisfy the condition

$$((id \otimes M) \circ (\pi^r \otimes id) \circ (Q_j^q \otimes S) \circ \pi^p)(v^p) = \sum_{k=1}^{d_q} Q_k^q(v^p) \otimes \pi_{kj}^q \tag{31}$$

for all $v^p \in V^p$ and all $j = 1, 2, \ldots, d_q$. Clearly this definition involves *only* quantities defined for $\mathcal{A}$ and its coactions. Both sides (31) are members of $V^r \otimes \mathcal{A}$. (The motivation behind the definition (31) is explained in Section B of the Appendix.)

It will now be shown that (31) provides a *consistent* definition, in that it can be re-expressed by saying that the operators Q_j^q (for $j = 1, 2, \ldots, d_q$) form the basis of an irreducible subspace of a carrier space for a certain right coaction of $\mathcal{A}$. This right coaction will be denoted by $\pi_{\mathcal{L}^{pr}}$, as its carrier space is $\mathcal{L}^{pr}$. The *definition* of $\pi_{\mathcal{L}^{pr}}$ is then that it is the mapping of $\pi_{\mathcal{L}^{pr}}$ into $\pi_{\mathcal{L}^{pr}} \otimes \mathcal{A}$ that specified by

$$\pi_{\mathcal{L}^{pr}}(Q) = \sum_{i,j=1}^{d_p} \sum_{m,n=1}^{d_r} (q_{ni}\, \mathcal{P}_{jm}^{pr}) \otimes M(\pi_{mn}^r \otimes S(\pi_{ij}^p)) \ , \tag{32}$$

for all $Q \in \mathcal{L}^{pr}$, where q_{ni} and $\mathcal{P}_{jm}^{pr}$ are defined in (17) and (14). (The motivation for the definition (32) is given in Section B of the Appendix.)

It is then quite easily shown that $\pi_{\mathcal{L}^{pr}}$ satisfies (5) and (6) (with π_V and V replaced by $\pi_{\mathcal{L}^{pr}}$ and $\mathcal{L}^{pr}$ respectively), and hence $\pi_{\mathcal{L}^{pr}}$ is indeed a right coaction with carrier space $\mathcal{L}^{pr}$. Moreover, it is easily demonstrated that

$$(\pi_{\mathcal{L}^{pr}}(Q))(v^p \otimes 1_{\mathcal{A}}) = ((id \otimes M) \circ (\pi^r \otimes id) \circ (Q \otimes S) \circ \pi^p)(v^p) \tag{33}$$

for all $v^p \in V^p$ and all $Q \in \mathcal{L}^{pr}$. Thus (31) and (33) imply that the definition (31) can be written equivalently as

$$\pi_{\mathcal{L}^{pr}}(Q^q_j) = \sum_{k=1}^{d_q} Q^q_k \otimes \pi^q_{kj} \tag{34}$$

(for all $j = 1, 2, \ldots, d_q$). Because (34) is similar in form to (7), and as $\pi_{\mathcal{L}^{pr}}$ is a right coaction with carrier space $\mathcal{L}^{pr}$, the consistency of the definition (31) is now ensured.

Now consider the situation in which V is a vector space that is a direct sum of carrier spaces of unitary irreducible corepresentations of $\mathcal{A}$ and which contains at least $V^p \oplus V^r$. Let π be the mapping of V into $V \otimes \mathcal{A}$ that coincides with π^p on V^p and with π^r on V^r, and which acts similarly on any other carrier spaces that might be contained in V. Then the generalization of (31) is clearly

$$((id \otimes M) \circ (\pi \otimes id) \circ (Q^q_j \otimes S) \circ \pi)(v) = \sum_{k=1}^{d_q} Q^q_k(v) \otimes \pi^q_{kj} \tag{35}$$

for all $v \in V$ and all $j = 1, 2, \ldots, d_q$. (The consistency of the definition (35) is an immediate consequence of the consistency of (31)).

2. Definition of the twisted irreducible tensor operators $\widetilde{Q}^q_j$

The *twisted irreducible tensor operators* $\widetilde{Q}^q_j$ belonging to the unitary irreducible right coaction π^q of $\mathcal{A}$ are *defined* to be members of $\mathcal{L}^{pr}$ that satisfy the condition

$$((id \otimes M) \circ (id \otimes \sigma) \circ (\pi^r \otimes id) \circ (\widetilde{Q}^{qR}_j \otimes S^{-1}) \circ \pi^p)(v^p) = \sum_{k=1}^{d_q} \widetilde{Q}^q_k(v^p) \otimes \pi^q_{kj} \tag{36}$$

for all $v^p \in V^p$ and all $j = 1, 2, \ldots, d_q$. This definition (36) differs from the corresponding definition (31) only in the replacement of M by $M \circ \sigma$ *and* S by S^{-1} (neither of which have

any effect in the special case in which $\mathcal{A}$ is commutative). (See Section B of the Appendix for further discussion of this pair of substitutions. It should be recorded that Rittenberg and Scheunert[13] noted previously, in the context of what was essentially the abstract carrier space formalism of Section I as generalized to irreducible *representations* of the dual $\mathcal{A}'$, that these substitutions do produce another type of irreducible tensor operator, but they did not pursue this observation.)

The demonstration that (36) provides a *consistent* definition again involves showing that it can be re-expressed by saying that the operators $\widetilde{Q}^q_j$ (for $j = 1, 2, \ldots, d_q$) form the basis of an irreducible subspace of a carrier space for another right coaction $\widetilde{\pi}_{\mathcal{L}^{pr}}$ of $\mathcal{A}$. This right coaction is *defined* as the mapping of $\mathcal{L}^{pr}$ into $\mathcal{L}^{pr} \otimes \mathcal{A}$ that specified by

$$\widetilde{\pi}_{\mathcal{L}^{pr}}(Q) = \sum_{i,j=1}^{d_p} \sum_{m,n=1}^{d_r} (q_{ni}\, \mathcal{P}^{pr}_{jm}) \otimes M(S^{-1}(\pi^p_{ij} \otimes \pi^r_{mn})) \tag{37}$$

for all $Q \in \mathcal{L}^{pr}$. (The motivation for the definition (37) is given in Section C of the Appendix). Then

$$(\widetilde{\pi}_{\mathcal{L}^{pr}}(Q))(v^p \otimes 1_{\mathcal{A}}) = ((id \otimes M) \circ (id \otimes \sigma) \circ (\pi^r \otimes id) \circ (Q \otimes S^{-1}) \circ \pi^p)(v^p) \tag{38}$$

for all $v^p \in V^p$ and all $Q \in \mathcal{L}^{pr}$. Thus (36) and (38) imply that the definition (36) can be written equivalently as

$$\widetilde{\pi}_{\mathcal{L}^{pr}}(\widetilde{Q}^{qR}_j) = \sum_{k=1}^{d_q} \widetilde{Q}^{qR}_k \otimes \pi^q_{kj} \tag{39}$$

(for all $j = 1, 2, \ldots, d_q$), which then ensures its consistency.

In the situation in which V is a vector space that is a direct sum of carrier spaces of unitary irreducible corepresentations of $\mathcal{A}$ and which contains at least $V^p \oplus V^r$, and with the mapping π from V into $V \otimes \mathcal{A}$ that is defined in the end of the previous subsubsection, the generalization of (36) is clearly

$$((id \otimes M) \circ (id \otimes \sigma) \circ (\pi \otimes id) \circ (\widetilde{Q}^{qR}_j \otimes S^{-1}) \circ \pi)(v) = \sum_{k=1}^{d_q} \widetilde{Q}^q_k(v) \otimes \pi^q_{kj} \tag{40}$$

for all $v \in V$ and all $j = 1, 2, \ldots, d_q$. (Again, the consistency of the definition (40) is an immediate consequence of the consistency of (36)).

C. Properties of irreducible tensor operators

1. The identity operator as an irreducible tensor operator

Suppose that V is a vector space that is a direct sum of carrier spaces of unitary irreducible corepresentations of $\mathcal{A}$ and which contains at least $V^p \oplus V^r$, and that π is the mapping of V into $V \otimes \mathcal{A}$ that is defined in the previous subsection. Suppose that Q is the *identity operator* id of V (so that $Q(v) = v$ for all $v \in V$). Then, on using (5) and (6), together with the Hopf algebra properties $M \circ (id \otimes S) \circ \Delta = u \circ \epsilon$ and $u(1_{\boldsymbol{C}}) = 1_{\mathcal{A}}$, it follows that

$$((id \otimes M) \circ (\pi \otimes id) \circ (id \otimes S) \circ \pi)(v) = v \otimes 1_{\mathcal{A}} \tag{41}$$

for all $v \in V$, which, by (35), leads to the conclusion that the identity operator id is an *ordinary* irreducible tensor operator for the one-dimensional *identity* corepresentation whose sole matrix coefficient is $1_{\mathcal{A}}$.

It is easily checked (using (40) in place of (35)), that id is also a *twisted* irreducible tensor operator for this identity corepresentation.

The same conclusions for identity operators follow directly from (31) and (36) in the special case in which $p = r$.

2. Two useful identities for the ordinary irreducible tensor operators Q^q_j and $\widetilde{Q}^q_j$

If Q^q_k is an *ordinary* irreducible tensor operator belonging to the unitary irreducible corepresentation π^q of $\mathcal{A}$ (as defined in (31)), and v^p_j (for $j = 1, 2, \ldots, d_p$) provides a basis for the carrier space V^p of the unitary irreducible corepresentation π^p of $\mathcal{A}$, then

$$\pi^r(Q^q_k(v^p_j)) = \sum_{s=1}^{d_p} \sum_{t=1}^{d_q} (Q^q_t(v^p_s)) \otimes (M(\pi^q_{tk} \otimes \pi^p_{sj})), \tag{42}$$

for all $j = 1, 2, \ldots, d_p$, and $k = 1, 2, \ldots, d_q$. By contrast, if $\widetilde{Q}^q_k$ is a *twisted* irreducible tensor operator belonging π^q, then

$$\pi^r(\widetilde{Q}_k^q(v_j^p)) = \sum_{s=1}^{d_p}\sum_{t=1}^{d_q}(\widetilde{Q}_t^q(v_s^p)) \otimes (M(\pi_{sj}^p \otimes \pi_{tk}^q)), \tag{43}$$

for all $j = 1, 2, \ldots, d_p$ and $k = 1, 2, \ldots, d_q$. It should be noted that the factors in the second term of the right-hand side of (43) are interchanged relative to those of (42).

The proof of (42) is as follows. On applying (7) and the relation $S(\pi_{ks}^p) = \pi_{sk}^{p*}$, the left-hand side of (31) (with $v^p = v_s^p$) becomes

$$((id \otimes M) \circ (\pi^r \otimes id))(\sum_{k=1}^{d_p}(Q_j^q(v_k^p)) \otimes \pi_{sk}^{p*})$$

On multiplying from the right with $id \otimes \pi_{si}^p$, summing over s, and applying the relation $M(\pi_{sk}^{p*} \otimes \pi_{si}^p) = \delta_{ik} 1_{\mathcal{A}}$, this reduces to $\pi^r(Q_i^q(v_j^p))$. However, multiplication of the right-hand side of (31) from the right with $id \otimes \pi_{si}^p$ and summing over s produces $\sum_{s=1}^{d_p}\sum_{t=1}^{d_q}(Q_t^q(v_s^p)) \otimes (M(\pi_{tk}^q \otimes \pi_{sj}^p))$. The line of proof for (43) is similar.

3. Identification of the corepresentations $\pi_{\mathcal{L}^{pr}}$ and $\widetilde{\pi}_{\mathcal{L}^{pr}}$ of $\mathcal{A}$

It is easily shown from the definition (32) of $\pi_{\mathcal{L}^{pr}}$ that

$$\pi_{\mathcal{L}^{pr}}(\mathcal{P}_{ij}^{pr}) = \sum_{m=1}^{d_p}\sum_{n=1}^{d_r} \mathcal{P}_{mn}^{pr} \otimes (\pi^r \boxtimes \bar{\pi}^p)_{nm,ji} \ , \tag{44}$$

where $\mathcal{P}_{ij}^{pr}$ are the operators defined in (14), and where $\bar{\pi}^p$ is the corepresentation of $\mathcal{A}$ that is *conjugate* to π^p, so that its matrix coefficients are given by $\bar{\pi}_{jk}^p = (\pi_{jk}^p)^*$. Then, by (20) and (22),

$$\pi_{\mathcal{L}^{pr}}(\mathcal{P}_{ij}^{pr}) = \sum_{m=1}^{d_p}\sum_{n=1}^{d_r} \mathcal{P}_{mn}^{pr} \otimes (\bar{\pi}^p \widetilde{\boxtimes} \pi^r)_{mn,ij} \ . \tag{45}$$

This shows that $\pi_{\mathcal{L}^{pr}}$ is actually the right coaction that is given by the *twisted* tensor product $\bar{\pi}^p \widetilde{\boxtimes} \pi^r$, and that the operators $\mathcal{P}_{ij}^{pr}$ are the basis vectors of the carrier space $\mathcal{L}^{pr}$ of this coaction.

Taken with (34), this indicates that the irreducible tensor operators Q_j^q of the definition (31) exist only if π^q is contained in the reduction of $\bar{\pi}^p \widetilde{\boxtimes} \pi^r$. As $\bar{\pi}^p \widetilde{\boxtimes} \pi^r$ and $\pi^r \boxtimes \bar{\pi}^p$ are equivalent[1], this implies that Q_j^q exists only if $n_{r\bar{p}}^q > 0$. But $n_{r\bar{p}}^q = n_{qp}^r$ (c.f. (I.125)), so Q_j^q

exists only if $n^r_{qp} > 0$. (These observations are confirmed by the explicit expressions for the irreducible tensor operators given in (47) below and by the Wigner-Eckart theorem of (50) below).

Similarly, one can show from the definition (37) of $\widetilde{\pi}_{\mathcal{L}^{pr}}$ that

$$\widetilde{\pi}_{\mathcal{L}^{pr}}(\mathcal{P}^{pr}_{ij}) = \sum_{m=1}^{d_p} \sum_{n=1}^{d_r} \mathcal{P}^{pr}_{mn} \otimes (\overline{\pi^{p\ddagger}} \boxtimes \pi^r)_{mn,ij} \ , \tag{46}$$

where $\overline{\pi^{p\ddagger}}$ is the corepresentation of $\mathcal{A}$ that is *conjugate* to the corepresentation $\pi^{p\ddagger}$ that is itself *doubly contragredient* to π^p. This shows that $\widetilde{\pi}_{\mathcal{L}^{pr}}$ is the right coaction that is given by the ordinary tensor product $\overline{\pi^{p\ddagger}} \boxtimes \pi^r$, and that the operators $\mathcal{P}^{pr}_{ij}$ are the basis vectors of the carrier space $\mathcal{L}^{pr}$ of this coaction.

Taken with (39), this shows that the twisted irreducible tensor operators $\widetilde{Q}^q_j$ of the definition (36) exist only if π^q is contained in the reduction of $\overline{\pi^{p\ddagger}} \boxtimes \pi^r$. As $\overline{\pi^{p\ddagger}}$ and $\bar{\pi}^p$ are equivalent[7–12], this implies that $\widetilde{Q}^q_j$ exists only if $n^q_{\bar{p}r} > 0$. But $n^q_{\bar{p}r} = n^r_{pq}$ (c.f. (I.125)), so $\widetilde{Q}^q_j$ exists only if $n^r_{pq} > 0$. (These observations are confirmed by the explicit expressions for the twisted irreducible tensor operators given in (49) below and by the Wigner-Eckart theorem of (52) below).

4. Explicit expressions for the irreducible tensor operators

If $n^r_{qp} > 0$ there exist n^r_{qp} linearly independent *ordinary* irreducible tensor operators that satisfy (31). These are given by

$$Q^{q,\alpha}_j = \sum_{i=1}^{d_p} \sum_{\ell=1}^{d_r} \left(\begin{array}{cc|cc} r & \bar{p} & q & , \ \alpha \\ \ell & i & j & \end{array} \right) \mathcal{P}^{pr}_{i\ell} \, , \tag{47}$$

for $\alpha = 1, 2, \ldots, n^r_{qp}$ and $j = 1, 2, \ldots, d_q$. Here the label $\bar{p}$ in the Clebsch-Gordan coefficients relates to the corepresentation $\bar{\pi}^p$ that is conjugate to π^p. As noted in the previous subsubsection, $n^q_{r\bar{p}} = n^r_{qp}$ (c.f. (I.125)).

The proof of (47) is as follows. The analogue of (I.130) for the tensor product $\pi^r \boxtimes \bar{\pi}^p$ is

$$\sum_{\ell=1}^{d_r}\sum_{i=1}^{d_p}(\pi^r\boxtimes\bar{\pi}^p)_{nm,\ell i}\left(\begin{array}{cc|c} r & \bar{p} & q\ ,\ \alpha \\ \ell & i & j \end{array}\right)=\sum_{k=1}^{d_q}\left(\begin{array}{cc|c} r & \bar{p} & q\ ,\ \alpha \\ n & m & k \end{array}\right)\pi^q_{kj} \tag{48}$$

for $m=1,2,\ldots,d_p$, $j=1,2,\ldots,d_q$, $n=1,2,\ldots,d_r$, and $\alpha=1,2,\ldots,n^r_{qp}$. However, by (32), (44), and (47),

$$\pi_{\mathcal{L}^{pr}}(Q^{q,\alpha}_j)=\sum_{i,m=1}^{d_p}\sum_{\ell,n=1}^{d_r}\left(\begin{array}{cc|c} r & \bar{p} & q\ ,\ \alpha \\ \ell & i & j \end{array}\right)\mathcal{P}^{pr}_{mn}\otimes(\pi^r\boxtimes\bar{\pi}^p)_{nm,\ell i}\ .$$

On applying (48) and (47), this reduces to

$$\pi_{\mathcal{L}^{pr}}(Q^{q,\alpha}_j)=\sum_{k=1}^{d_q}Q^{q,\alpha}_k\otimes\pi^q_{kj}$$

(for all $j=1,2,\ldots,d_q$ and $\alpha=1,2,\ldots,n^r_{qp}$). That is, the operators $Q^{q,\alpha}_j$ defined in (47) satisfy (34), which is equivalent to (31).

Similarly, if $n^r_{pq}>0$ there exist n^r_{pq} linearly independent *twisted* irreducible tensor operators that satisfy (31). These are given by

$$\widetilde{Q}^{q,\alpha}_j=\sum_{i=1}^{d_p}\sum_{\ell=1}^{d_r}\left(\begin{array}{cc|c} \overline{p^\ddagger} & r & q\ ,\ \alpha \\ i & \ell & j \end{array}\right)\mathcal{P}^{pr}_{i\ell}\,, \tag{49}$$

for $\alpha=1,2,\ldots,n^r_{pq}$ and $j=1,2,\ldots,d_q$. Here the label $\overline{p^\ddagger}$ in the Clebsch-Gordan coefficients relates to the corepresentation $\overline{\pi^{p\ddagger}}$ that is conjugate to the corepresentation $\pi^{p\ddagger}$ that is itself doubly contragredient to π^p. It should be noted that $n^q_{\bar{p}r}=n^r_{pq}=n^q_{\overline{p^\ddagger}r}$ (c.f. (I.125)).

The proof of (49) is similar to that of (47), but uses the right coaction $\widetilde{\pi}_{\mathcal{L}^{pr}}$ in the place of $\pi_{\mathcal{L}^{pr}}$. Also needed is the relation

$$(\overline{\pi^{p\ddagger}})_{ij}=\sum_{k,\ell=1}^{d_p}\overline{F^p_{ik}}\ (\bar{\pi}^p)_{k\ell}\ \overline{((\mathbf{F}^p)^{-1})_{\ell j}}\ ,$$

which follows from (30), and the corresponding relation

$$\left(\begin{array}{cc|c} \bar{p} & r & q\ ,\ \alpha \\ i & \ell & j \end{array}\right)=\sum_{k=1}^{d_p}\overline{((\mathbf{F}^p)^{-1})_{ik}}\ \left(\begin{array}{cc|c} \overline{p^\ddagger} & r & q\ ,\ \alpha \\ k & \ell & j \end{array}\right)\ .$$

V. THEOREMS OF THE WIGNER-ECKART TYPE

If π^p, π^q, and π^r are unitary irreducible corepresentations of $\mathcal{A}$ of dimensions d_p, d_q, and d_r respectively, v^p_j and v^r_ℓ are basis vectors belonging to the carrier spaces V^p and V^r of π^p and π^r respectively, and Q^q_k is an *ordinary* irreducible tensor operator belonging to π^q (as defined in (31)), then

$$\langle v^r_\ell, Q^q_k(v^p_j)\rangle = \sum_{\alpha=1}^{n^r_{qp}} \left(\begin{array}{cc|c} r & , \alpha & q \ p \\ \ell & & k \ j \end{array} \right) (r \mid Q^q \mid p)_\alpha \ , \tag{50}$$

for all $j = 1, 2, \ldots, d_p$, all $k = 1, 2, \ldots, d_q$, and all $\ell = 1, 2, \ldots, d_r$. Here the *reduced matrix elements* $(r \mid Q^q \mid p)_\alpha$ are given by

$$\begin{aligned} &(r \mid Q^q \mid p)_\alpha \\ &= \textstyle\sum_{s=1}^{d_p} \sum_{t=1}^{d_q} \sum_{u,v=1}^{d_r} \langle v^r_u, Q^q_t(v^p_s)\rangle \left(\begin{array}{c|cc} q \ p & r & , \alpha \\ t \ s & v & \end{array} \right) \{((\mathbf{F}^r)^{-1})_{vu}/tr((\mathbf{F}^r)^{-1})\} \end{aligned} \tag{51}$$

for $\alpha = 1, 2, \ldots, n^r_{qp}$, and $\langle \ , \ \rangle$ denotes the inner product of V^r. Here $\mathbf{F}^r$ is the matrix defined in (30).

On the other hand, if $\widetilde{Q}^q_k$ is a *twisted* irreducible tensor operator belonging to π^q (as defined in (36)), then

$$\langle v^r_\ell, \widetilde{Q}^q_k(v^p_j)\rangle = \sum_{\alpha=1}^{n^r_{pq}} \left(\begin{array}{cc|c} r & , \alpha & p \ q \\ \ell & & j \ k \end{array} \right) (r \mid \widetilde{Q}^q \mid p)_\alpha, \tag{52}$$

for all $j = 1, 2, \ldots, d_p$, all $k = 1, 2, \ldots, d_q$, and all $\ell = 1, 2, \ldots, d_r$, where the reduced matrix elements $(r \mid \widetilde{Q}^q \mid p)_\alpha$ are given by

$$\begin{aligned} &(r \mid \widetilde{Q}^q \mid p)_\alpha \\ &= \textstyle\sum_{s=1}^{d_p} \sum_{t=1}^{d_q} \sum_{u,v=1}^{d_r} \langle v^r_u, \widetilde{Q}^q_t(v^p_s)\rangle \left(\begin{array}{c|cc} p \ q & r & , \alpha \\ s \ t & v & \end{array} \right) \{((\mathbf{F}^r)^{-1})_{vu}/tr((\mathbf{F}^r)^{-1})\} \end{aligned} \tag{53}$$

for $\alpha = 1, 2, \ldots, n^r_{pq}$.

The results (50) and (52) again exhibit the classic Wigner-Eckart theorem behaviour, in that they show that the j, k, and ℓ dependences of the inner products $\langle v^r_\ell, Q^q_k(v^p_j)\rangle$ and

$\langle v_\ell^r, \widetilde{Q}_k^q(v_j^p)\rangle$ are determined only by Clebsch-Gordan coefficients, but it should be noted that in the general case in which $\mathcal{A}$ is non-commutative, the inner products for the *ordinary* and *twisted* irreducible tensor operators involve *different* sets of Clebsch-Gordan coefficients.

The proof of (50) is as follows. The condition for a corepresentation π_V of $\mathcal{A}$ to be unitary is that

$$\sum_{[v]}\langle w, v_{[1]}, \rangle S(v_{[2]}) = \sum_{[w]}\langle w_{[1]}, v\rangle (w_{[2]})^* \tag{54}$$

for all $v, w \in V$, the carrier space of π_V, where

$$\pi_V(v) = \sum_{[v]} v_{[1]} \otimes v_{[2]} \ , \tag{55}$$

with $v_{[1]} \in V$ and $v_{[2]} \in \mathcal{A}$ (c.f. (I.51)). Thus with $v = Q_j^q(v_i^p)$ and $w = v_\ell^r$, (54), (55), (42), and (7) imply that

$$\sum_{s=1}^{d_p}\sum_{t=1}^{d_q}\langle v_\ell^r, Q_t^q(v_s^p)\rangle \ S(M(\pi_{tj}^q \otimes \pi_{si}^p)) = \sum_{u=1}^{d_r}\langle v_u^r, Q_j^q(v_i^p)\rangle (\pi_{u\ell}^r)^* \ . \tag{56}$$

But $(\pi_{u\ell}^r)^* = S(\pi_{\ell u}^r)$, so acting on both sides with S^{-1} (which is well defined for a compact quantum group algebra[7−12]), multiplying through from the left by $(\pi_{\ell k}^r)^*$, summing over ℓ, and using the relation $\sum_\ell M((\pi_{\ell k}^r)^* \otimes (\pi_{\ell u}^r)) = \delta_{uk}1_{\mathcal{A}}$, (56) reduces to

$$\sum_{\ell=1}^{d_r}\sum_{s=1}^{d_p}\sum_{t=1}^{d_q}\langle v_\ell^r, Q_t^q(v_s^p)\rangle \ ((\pi_{\ell k}^r)^* \pi_{tj}^q \otimes \pi_{si}^p) = \langle v_k^r, Q_j^q(v_i^p)\rangle 1_{\mathcal{A}} \ . \tag{57}$$

On acting with the Haar functional h, and applying (29) and the relation $h(1_{\mathcal{A}}) = 1$, (50) follows immediately. The proof of (52) is similar.

VI. EXAMPLE: IRREDUCIBLE TENSOR OPERATORS FOR THE STANDARD DEFORMATION OF THE FUNCTION ALGEBRA OF THE COMPACT LIE GROUP $SU(2)$

It is particularly interesting to study the foregoing theory for the case in which $\mathcal{A}$ is the standard deformation of the function algebra of the compact Lie group $SU(2)$, because both

this Hopf algebra $\mathcal{A}$ and its dual $\mathcal{A}'$ have been very extensively investigated, the former in the language of 'compact matrix pseudogroups' and the latter as the deformation $U_q(sl(2))$ of the universal enveloping algebra $U(sl(2))$ of the simple Lie algebra $sl(2)$.

A. Structure of the standard deformation of the function algebra of the compact Lie group $SU(2)$

It is well known (c.f. Ref. 16) that the irreducible representations of the deformation $U_q(sl(2))$ (for generic q) can be labelled by a single index j, which takes values $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, the irreducible representation corresponding to j being $(2j+1)$-dimensional, with rows and columns that may be labeled by indices m' and m that take values $-j, -j+1, \ldots, j-1, j$, exactly as for the simple Lie algebra $sl(2)$. To each of these representations corresponds a *co*representation of the dual Hopf algebra $\mathcal{A}$. Consequently the labels for corepresentations of $\mathcal{A}$ will henceforth always be denoted by j (possibly with a prime or subscript attached) and the rows and columns of the corresponding matrix coefficients will be labelled by these indices m and m' (possibly with subscripts attached). (Although q was used in all the other sections of this paper to indicate an irreducible representation or corepresentation, in this section it will be employed to denote the standard deformation parameter).

All of the irreducible corepresentations π^j (for $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$) may be taken to be unitary, and their matrix coefficients $\pi^j_{m'm}$ form a basis for $\mathcal{A}$. Moreover *every* matrix coefficient $\pi^j_{m'm}$ for $j \geq 1$ can be written as a polynomial in the matrix coefficients of $\pi^{1/2}$, while $\pi^0_{00} = 1_{\mathcal{A}}$. Let

$$\boldsymbol{\pi}^{1/2} = \begin{pmatrix} X & U \\ V & Y \end{pmatrix} , \tag{58}$$

where the entries are assumed to satisfy the relations

$$\left.\begin{aligned} &XU = q^{-1}UX,\ XV = q^{-1}VX,\ YU = qUY,\ YV = qVY, \\ &UV = VU,\ XY - q^{-1}UV = 1_{\mathcal{A}},\ YX - qUV = 1_{\mathcal{A}}. \end{aligned}\right\} \tag{59}$$

In the language of 'matrix pseudogroups' the matrix coefficients $\pi^j_{m'm}$ are called 'quantum d-functions' and are denoted by $d^j_{m'm}$. The work of Nomura[15] then implies that

$$\begin{aligned}\pi^j_{m'm} &= q^{(m'-m)(2j-m'+m)/2}\{[j+m']![j-m']![j+m]![j-m]!\} \\ &\times \sum_a \frac{q^{a(2j-m'+m-a)}X^{j+m-a}U^{m'-m+a}V^aY^{j-m'-a}}{[a]![j+m-a]![m'-m+a]![j-m'-a]!}\,,\end{aligned} \tag{60}$$

where $[n]=(q^n-q^{-n})/(q-q^{-1})$ and $[n]!=[n][n-1][n-2]\dots[2][1]$, and where the sum over a is over all integers such that the expressions in the q- factorials are non-negative. (The present quantity q is actually $q^{1/2}$ in the notation of Nomura[15,16]). Then, for example

$$\boldsymbol{\pi}^1 = \begin{pmatrix} X^2 & q^{1/2}[2]^{1/2}XU & U^2 \\ q^{1/2}[2]^{1/2}XV & XY+qUV & q^{1/2}[2]^{1/2}UY \\ V^2 & q^{1/2}[2]^{1/2}VY & Y^2 \end{pmatrix}\,, \tag{61}$$

and

$$\boldsymbol{\pi}^{3/2} = \begin{pmatrix} X^3 & q[3]^{1/2}X^2U & q[3]^{1/2}XU^2 & U^3 \\ q[3]^{1/2}X^2V & X^2Y+q^2[2]XUV & q[2]XUY+q^2U^2V & q[3]^{1/2}U^2Y \\ q[3]^{1/2}XV^2 & q[2]XVY+q^2UV^2 & XY^2+q^2[2]UVY & q[3]^{1/2}UY^2 \\ V^3 & q[3]^{1/2}V^2Y & q[3]^{1/2}VY^2 & Y^3 \end{pmatrix}\,. \tag{62}$$

The product of any two matrix coefficients can (at least in principle) be deduced from the expressions.

An alternative way of getting the product of any two matrix coefficients is to invoke (28), for the Clebsch-Gordan coefficients are known for this $\mathcal{A}$. Indeed for this $\mathcal{A}$ the Clebsch-Gordan coefficients exhibit two simplifying features. Firstly, the multiplicity is always just 1, so the index α in the Clebsch-Gordan coefficients of (23) may be omitted, and secondly, the Clebsch-Gordan coefficients can be taken to be purely real. As the Clebsch-Gordan series for $\pi^{j_1}\boxtimes\pi^{j_2}$ is the direct sum of π^j with $j=j_1+j_2, j_1+j_2-1,\dots,|j_1-j_2|$, (28) reduces in this case to

$$M(\pi^{j_1}_{m_1'm_1}\otimes\pi^{j_2}_{m_2'm_2}) = \sum_{j=|j_1-j_2|}^{j_1+j_2}\ \sum_{m',m=-j}^{j} \left(\begin{array}{cc|c} j_1 & j_2 & j \\ m_1' & m_2' & m' \end{array}\right)\left(\begin{array}{cc|c} j_1 & j_2 & j \\ m_1 & m_2 & m \end{array}\right)\pi^j_{m'm}\,. \tag{63}$$

Various equivalent expressions for the Clebsch-Gordan coefficients appear in the literature, but the most convenient for application here is that given by Nomura[16], which, in the present notation, is

$$\left(\begin{array}{cc|c} j_1 & j_2 & j \\ m_1 & m_2 & m \end{array}\right) =$$

$$\Delta(j_1,j_2,j)q^{\{x(j_1)+x(j_2)-x(j)+2(j_1j_2+j_1m_2-j_2m_1)\}/2} \tag{64}$$
$$\times\{[j_1+m_1]![j_1-m_1]![j_2+m_2]![j_2-m_2]![j+m]![j-m]![2j+1]\}^{1/2}$$
$$\times\sum_a \frac{(-1)^a q^{-a(j_1+j_2+j+1)/2}}{[a]![j_1+j_2-j-a]![j_1-m_1-a]![j_2+m_2-a]![j-j_2+m_1+a]![j-j_1-m_2+a]!}\ ,$$

where $\Delta(a,b,c) = \{[-a+b+c]![a-b+c]![a+b-c]!/[a+b+c+1]!\}^{1/2}$ and $x(a) = a(a+1)$, and where the sum over a is over all integers such that the expressions in the q-factorials are non-negative. In particular

$$\left(\begin{array}{cc|c} j+\frac{1}{2} & j & \frac{1}{2} \\ m+\frac{1}{2} & -m & \frac{1}{2} \end{array}\right) = (-1)^{j-m}\, q^{-j+\frac{1}{2}+\frac{3}{2}m}\, [j+m+1]^{1/2}\, \{[2][2j]!/[2j+2]!\}^{1/2}\ \ , \tag{65}$$

and

$$\left(\begin{array}{cc|c} j+\frac{1}{2} & j & \frac{1}{2} \\ m-\frac{1}{2} & -m & -\frac{1}{2} \end{array}\right) = (-1)^{j-m}\, q^{\frac{1}{2}+\frac{3}{2}m}\, [j-m+1]^{1/2}\, \{[2][2j]!/[2j+2]!\}^{1/2}\ \ . \tag{66}$$

The action of the coproduct Δ and counit ϵ of $\mathcal{A}$ on the generators of $\mathcal{A}$ is given by

$$\left.\begin{array}{l} \Delta(X) = X\otimes X + U\otimes V,\ \ \Delta(Y) = V\otimes U + Y\otimes Y, \\ \Delta(U) = X\otimes U + U\otimes Y,\ \ \Delta(V) = V\otimes X + Y\otimes V, \end{array}\right\}$$

and

$$\epsilon(X) = 1, \epsilon(Y) = 1, \epsilon(U) = 0, \epsilon(V) = 0.$$

Moreover, the action of the star-operation $*$ of $\mathcal{A}$ on the generators of $\mathcal{A}$ may be taken to be

$$X^* = Y,\ Y^* = X,\ U^* = -q^{-1}V,\ V^* = -qU\ , \tag{67}$$

which implies[15] that its action on any matrix coefficient is given by

$$(\pi^j_{m'm})^* = (-1)^{m-m'} q^{m-m'} \pi^j_{-m',-m} \ . \tag{68}$$

As $S(\pi^j_{m'm}) = (\pi^j_{mm'})^*$ (c.f. (I.52)), it follows that

$$S(\pi^j_{m'm}) = (-1)^{-(m-m')} q^{-(m-m')} \pi^j_{-m,-m'} \ . \tag{69}$$

Thus

$$S^2(\pi^j_{m'm}) = q^{-2(m-m')} \pi^j_{m'm} \ . \tag{70}$$

As the matrix coefficients of the doubly contragredient corepresentation $\pi^{j\ddagger}$ are given by $(\pi^j_{m'm})^\ddagger = S^2\pi^j_{m'm}$ (c.f. (I.57)), it follows that the $(2j+1)\times(2j+1)$ matrix of (30) (which appears in the expressions for the reduced matrix elements (51) and (53) of the Wigner-Eckart type theorems) is *diagonal* and that its elements are given by

$$F^j_{m'm} = \delta_{m'm} q^{-2(j-m)} \ . \tag{71}$$

B. Bosonic creation and annihilation operators as irreducible tensor operators

Let $b_1^\dagger, b_1$ and $b_2^\dagger, b_2$ be two pairs of 'deformed' bosonic creation and annihilation operators and N_1 and N_2 the associated number operators whose action on the infinite-dimensional Fock space spanned by the occupation number vectors $|n_i\rangle$ is given (c.f. Refs. 17,18) by

$$\left.\begin{aligned} b_i^\dagger|n_i\rangle &= [n_i+1]^{1/2}\,|n_i+1\rangle \ , \\ b_i|n_i\rangle &= [n_i]^{1/2}\,|n_i-1\rangle \ , \\ N_i|n_i\rangle &= n_i\,|n_i\rangle \ , \end{aligned}\right\} \tag{72}$$

for $i=1,2$, where it is assumed that the vacuum state vectors $|0\rangle$ are such that $b_i|0\rangle = 0$ for $i=1,2$. It is also assumed that every member of the set $\{b_1^\dagger, b_1, N_1\}$ commutes with every member of the set $\{b_2^\dagger, b_2, N_2\}$. In the deformed generalization of the Jordan-Schwinger realization of $sl(2)$ (c.f. Refs. 17,18), the basis vectors of the carrier spaces of the irreducible representations of $U_q(sl(2))$ are given by

$$v^j_m = |j+m, j-m\rangle \ , \tag{73}$$

and these, of course, are also the basis vectors of the carrier spaces of the irreducible *co*representations of $\mathcal{A}$.

Then

$$Q^{1/2}_{1/2} = b^\dagger_1 q^{-\frac{1}{2}N_2},\ Q^{1/2}_{-1/2} = b^\dagger_2 q^{\frac{1}{2}N_1}, \tag{74}$$

and

$$Q^{1/2}_{1/2} = q b_2 q^{\frac{1}{2}N_1},\ Q^{1/2}_{-1/2} = -b_1 q^{-\frac{1}{2}N_2}, \tag{75}$$

are two sets of pairs of *ordinary* irreducible tensor operators that belong to the 2-dimensional irreducible corepresentation $\pi^{1/2}$ of $\mathcal{A}$.

This will now be demonstrated for the the *first* pair (74), starting from the definition (35), and taking V to be the direct sum of all the carrier spaces of all the irreducible corepresentations of $\mathcal{A}$ (with just one such irreducible corepresentation being included from each equivalence class). Define the right coaction π of $\mathcal{A}$ by

$$\pi(v^j_m) = \sum_{m'=-j}^{j} v^j_{m'} \otimes \pi^j_{m'm} \ , \tag{76}$$

for all $j = 0, \frac{1}{2}, 1, \ldots$ and $m = j, j-1, \ldots, -j$. Then in this case (35) becomes

$$((id \otimes M) \circ (\pi \otimes id) \circ (Q^{1/2}_k \otimes S) \circ \pi)(v^j_m) = \sum_{\ell=-1/2}^{1/2} Q^{1/2}_\ell(v^j_m) \otimes \pi^{1/2}_{\ell k} \ , \tag{77}$$

for all $j = 0, \frac{1}{2}, 1, \ldots$ and $m = j, j-1, \ldots, -j$. It will now be shown that this is indeed satisfied for $k = \frac{1}{2}$. (The proof for $k = -\frac{1}{2}$ is similar). By (72), (69), (74), and (76), the left-hand side of (77) for $k = \frac{1}{2}$ is

$$\begin{array}{l} \sum_{m''=-j-1}^{j} \ \sum_{m'=-j}^{j} \ [j+m'+1]^{1/2} \, q^{-\frac{1}{2}(j-m')-(m-m')} \, (-1)^{-(m-m')} \\ \times \ v^{j+\frac{1}{2}}_{m''+\frac{1}{2}} \otimes M(\pi^{j+\frac{1}{2}}_{m''+\frac{1}{2}, m'+\frac{1}{2}} \otimes \pi^j_{-m,-m'}) \ . \end{array} \tag{78}$$

Similarly, by (72), (74), and (76), the right- hand side of (77) for $k = \frac{1}{2}$ is

$$[j+m+1]^{1/2} \, q^{-\frac{1}{2}(j-m)} \ v^{j+\frac{1}{2}}_{m+\frac{1}{2}} \otimes \pi^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}} + [j-m+1]^{1/2} \, q^{\frac{1}{2}(j+m)} \ v^{j+\frac{1}{2}}_{m-\frac{1}{2}} \otimes \pi^{\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}} \ , \tag{79}$$

so it remains to show that (78) reduces to (79). However, by (63) and (65), (78) reduces to

$$\begin{aligned}&\textstyle\sum_{m''=-j-1}^{j}\ \sum_{m'=-j}^{j}\ \sum_{j'=\frac{1}{2}}^{2j+\frac{1}{2}}\ \{[2][2j]!/[2j+2]!\}^{1/2}\,q^{(\frac{1}{2}j-m-\frac{1}{2})}\,(-1)^{(j-m)}\\ &\times\left(\begin{array}{cc|c} j+\frac{1}{2} & j & \frac{1}{2}\\ m'+\frac{1}{2} & -m' & \frac{1}{2}\end{array}\right)\left(\begin{array}{cc|c} j+\frac{1}{2} & j & j'\\ m''+\frac{1}{2} & -m & m''+\frac{1}{2}-m\end{array}\right)\\ &\times\left(\begin{array}{cc|c} j+\frac{1}{2} & j & j'\\ m'+\frac{1}{2} & -m' & \frac{1}{2}\end{array}\right)\,v^{j+\frac{1}{2}}_{m''+\frac{1}{2}}\otimes\pi^{j'}_{m''+\frac{1}{2}-m,\frac{1}{2}}\,.\end{aligned}\tag{80}$$

On invoking the Clebsch-Gordan orthogonality relation

$$\textstyle\sum_{m'=-j}^{j}\ \left(\begin{array}{cc|c} j+\frac{1}{2} & j & \frac{1}{2}\\ m'+\frac{1}{2} & -m' & \frac{1}{2}\end{array}\right)\left(\begin{array}{cc|c} j+\frac{1}{2} & j & j'\\ m'+\frac{1}{2} & -m' & \frac{1}{2}\end{array}\right)=\begin{cases}1, \text{ if } j'=\frac{1}{2}\\ 0, \text{ if } j'\neq\frac{1}{2}\end{cases}\,,\tag{81}$$

(80) (and hence (78)) reduces to

$$\begin{aligned}&\textstyle\sum_{m''=-j-1}^{j}\ \{[2][2j]!/[2j+2]!\}^{1/2}\,q^{(\frac{1}{2}j-m-\frac{1}{2})}\,(-1)^{(j-m)}\\ &\times\left(\begin{array}{cc|c} j+\frac{1}{2} & j & \frac{1}{2}\\ m''+\frac{1}{2} & -m & m''+\frac{1}{2}-m\end{array}\right)\,v^{j+\frac{1}{2}}_{m''+\frac{1}{2}}\otimes\pi^{j'}_{m''+\frac{1}{2}-m,\frac{1}{2}}\,.\end{aligned}\tag{82}$$

However, the remaining Clebsch-Gordan coefficients are zero if $m''+\frac{1}{2}-m>\frac{1}{2}$, i.e. if $m''>m$, and are zero if $m''+\frac{1}{2}-m<-\frac{1}{2}$, i.e. if $m''<m-1$, so these Clebsch-Gordan coefficients are non-zero only for $m''=m,m-1$. Thus (82) (and hence (78)) becomes

$$\begin{aligned}&\{[2][2j]!/[2j+2]!\}^{1/2}\,q^{(\frac{1}{2}j-m-\frac{1}{2})}\,(-1)^{(j-m)}\\ &\times\left\{\left(\begin{array}{cc|c} j+\frac{1}{2} & j & \frac{1}{2}\\ m-\frac{1}{2} & -m & -\frac{1}{2}\end{array}\right)\,v^{j+\frac{1}{2}}_{m-\frac{1}{2}}\otimes\pi^{\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}}+\left(\begin{array}{cc|c} j+\frac{1}{2} & j & \frac{1}{2}\\ m+\frac{1}{2} & -m & \frac{1}{2}\end{array}\right)\,v^{j+\frac{1}{2}}_{m+\frac{1}{2}}\otimes\pi^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}\right\},\end{aligned}$$

which, by (65) and (66), reduces to (79). Similarly

$$\widetilde{Q}^{1/2}_{1/2}=b_1^\dagger q^{\frac{1}{2}N_2},\ \widetilde{Q}^{1/2}_{-1/2}=b_2^\dagger q^{-\frac{1}{2}N_1},\tag{83}$$

and

$$\widetilde{Q}^{1/2}_{1/2}=q^{-1}b_2q^{-\frac{1}{2}N_1},\ \widetilde{Q}^{1/2}_{-1/2}=-b_1q^{\frac{1}{2}N_2},\tag{84}$$

are two sets of pairs of *twisted* irreducible tensor operators belonging to the 2-dimensional irreducible corepresentation $\pi^{1/2}$ of $\mathcal{A}$. (This is easily deduced from (78) and (79), because

in the special case of this algebra $\mathcal{A}$, (59) and (69) imply that the substitutions $M \to M \circ \sigma$ and $S \to S^{-1}$ merely correspond to the replacement of q by q^{-1}).

It has been observed previously by Biedenharn and Tarlini[19] that (78) provide a pair of irreducible tensor operators for the 2-dimensional irreducible representation of $U_q(sl(2))$, their argument essentially using (B10) and the generalized Jordan-Schwinger realization of the generators of $U_q(sl(2))$, together with various identities involving the creation and annihilation operators. The object of the above analysis in this subsection is to give an explicit demonstration of the applicability of the *new* definitions (B10) and (B11) for $\mathcal{A}$, which, of course, apply not merely to this example but to *any* compact quantum group algebra.

ACKNOWLEDGMENTS

This work was started while the author was on research leave at the Sektion Physik of the Ludwig-Maximilians-Universität, München. The author is grateful to Professor J. Wess and his colleagues for their hospitality and to the Deutscher Akademischer Austauschdienst for its financial support.

APPENDIX A: INTRODUCTION

The purpose of this Appendix is to *motivate* the definitions that are given in the main body of the paper for the irreducible tensor operators. This will be done by considering the simple special case in which the Hopf algebra $\mathcal{A}$ is the set of functions defined on a *finite* group $\mathcal{G}$ of order g, so that the dual $\mathcal{A}'$ of $\mathcal{A}$ is the group algebra of $\mathcal{G}$. Of course, as $\mathcal{A}$ is commutative in this special case, the resulting expressions are to some extent ambiguous, in that in this special case M is indistinguishable from $M \circ \sigma$ and S is indistinguishable from S^{-1}. The demonstration of the correctness, consistency, and usefulness of the definitions that are actually employed for the *general* case are the subject matter of the self-contained arguments of the main body of this paper.

A summary of the basic facts concerning the relationship of $\mathcal{A}$ and $\mathcal{A}'$ may be found in the Introduction to the Appendix of Paper I.

APPENDIX B: MOTIVATION FOR DEFINITIONS OF IRREDUCIBLE TENSOR OPERATORS

The starting point of the present argument is (2), which of course also applies to finite groups, and which may be rewritten as

$$\hat{\pi}'^r(x)\, Q^q_j\, \hat{\pi}'^p(x^{-1}) = \sum_{k=1}^{d_q} \Gamma^q(x)_{kj} Q^q_k \tag{B1}$$

for all $x \in \mathcal{G}$ and all $j = 1, 2, \ldots, d_q$. Here the operators $\hat{\pi}'^p(x)$ are defined by

$$\hat{\pi}'^p(x)(v^p_j) = \sum_{k=1}^{d_p} \Gamma^p(x)_{kj} v^p_k \ , \tag{B2}$$

and are related to the corresponding left action π'^p of $\mathcal{A}'$ (a mapping of the carrier space V^p into $V^p \otimes \mathcal{A}'$) by the prescription

$$\hat{\pi}'^p(x)(v^p_j) = \pi'^p(x \otimes v^p_j) \ . \tag{B3}$$

As here $S_{\mathcal{A}'}(x) = x^{-1}$ and $\Delta_{\mathcal{A}'}(x) = x \otimes x$, (B1) can be rewritten in purely Hopf algebraic terms (for $\mathcal{A}'$) as

$$(\pi'^r \circ (id \otimes Q^q_j) \circ (id \otimes \pi'^p) \circ (id \otimes S_{\mathcal{A}'} \otimes id) \circ (\Delta_{\mathcal{A}'} \otimes id))(x \otimes v^p) = \sum_{k=1}^{d_q} \Gamma^q(x)_{kj} Q^q_k(v^p) \tag{B4}$$

for all $x \in \mathcal{A}'$, all $v^p \in V^p$, and all $j = 1, 2, \ldots, d_q$. (Here the $\Gamma^q(x)_{kj}$ are now matrix elements of the irreducible representation π'^q of the Hopf algebra $\mathcal{A}'$).

This condition can be recast entirely in terms of quantities defined for the Hopf algebra $\mathcal{A}$ in the following way. As noted in equation (I.A7), the relationship between a right coaction π_V of $\mathcal{A}$ and the corresponding left action π'_V of $\mathcal{A}'$ (with the same carrier space V) is

$$\pi'_V(a' \otimes v) = (M_{V,\boldsymbol{C}} \circ (id \otimes ev) \circ (\sigma \otimes id) \circ (id \otimes \pi_V))(a' \otimes v) \tag{B5}$$

for all $a' \in \mathcal{A}'$ and all $v \in V$, where the evaluation map ev (from $\mathcal{A}' \otimes \mathcal{A}$ to $\mathbb{C}$) is defined (c.f. (I.41)) by

$$ev(a' \otimes a) = \langle a', a \rangle \tag{B6}$$

for all $a' \in \mathcal{A}'$ and all $a \in \mathcal{A}$. On applying this twice (once with with $\pi'_V = \pi'^r$ and once with $\pi'_V = \pi'^p$), the left-hand side of (B4) becomes

$$\begin{aligned}&(M_{V^r,\boldsymbol{C}} \circ (id \otimes ev) \circ (\sigma \otimes id) \circ (id \otimes \pi^r) \circ (id \otimes Q^q_j) \circ (id \otimes \pi^p) \circ (id \otimes M_{V^p,\boldsymbol{C}})\\ &\circ (id \otimes id \otimes ev) \circ (id \otimes \sigma \otimes id) \circ (id \otimes S_{\mathcal{A}'} \otimes \pi^p) \circ (\Delta_{\mathcal{A}'} \otimes id))(x \otimes v^p)\ .\end{aligned} \tag{B7}$$

As (I.A2) and (I.A3) can be rewritten as

$$(M_{\boldsymbol{C}} \circ (ev \otimes ev) \circ (id \otimes \sigma \otimes id) \circ (\Delta_{\mathcal{A}'} \otimes id))(a' \otimes a \otimes b) = (ev \circ (id \otimes M))(a' \otimes a \otimes b)$$

for all $a, b \in \mathcal{A}$ and all $a' \in \mathcal{A}'$, and

$$(ev \circ (S_{\mathcal{A}'} \otimes id))(a' \otimes a) = (ev \circ (id \otimes S))(a' \otimes a)$$

for all $a \in \mathcal{A}$ and all $a' \in \mathcal{A}'$, (B7) can be re-expressed as

$$\begin{aligned}&(M_{V^r,\boldsymbol{C}} \circ (id \otimes ev) \circ (\sigma \otimes id) \circ (id \otimes id \otimes M) \circ (id \otimes \pi^r \otimes id) \circ (id \otimes Q^q_j \otimes S)\\ &\circ (id \otimes \pi^p))(x \otimes v^p)\ .\end{aligned} \tag{B8}$$

However, the right-hand side of (B4) can be rewritten using (I.A10) as $\sum_{k=1}^{d_q} \langle x, \pi^q_{kj} \rangle Q^q_k(v^p)$, and hence as

$$\sum_{k=1}^{d_q} (M_{V^r,\boldsymbol{C}} \circ (id \otimes ev) \circ (\sigma \otimes id) \circ (id \otimes Q^q_k \otimes id))(x \otimes v^p \otimes \pi^q_{kj})\ . \tag{B9}$$

On equating (B8) and (B9), as the first three terms are common to both expressions, they can be removed. The remaining terms act simply as the identity on the factor x, so on removing this now trivial effect on x, it follows that (B4) is equivalent to

$$((id \otimes M) \circ (\pi^r \otimes id) \circ (Q^q_j \otimes S) \circ \pi^p)(v^p) = \sum_{k=1}^{d_q} Q^q_k(v^p) \otimes \pi^q_{kj}$$

for all $v^p \in V^p$ and all $j = 1, 2, \ldots, d_q$. As this involves *only* quantities defined for $\mathcal{A}$, it provides the desired criterion (31).

Now consider the situation in which V is a vector space that is a direct sum of carrier spaces of unitary irreducible corepresentations of $\mathcal{A}$ and which contains at least $V^p \oplus V^r$. Let π be the mapping of V into $V \otimes \mathcal{A}$ that coincides with π^p on V^p and with π^r on V^r, and which acts similarly on any other carrier spaces that might be contained in V. Of course V is also is the direct sum of carrier spaces of unitary irreducible representations of $\mathcal{A}'$. Then the generalization of (B4) to this situation is

$$(\pi' \circ (id \otimes Q_j^q) \circ (id \otimes \pi') \circ (id \otimes S_{\mathcal{A}'} \otimes id) \circ (\Delta_{\mathcal{A}'} \otimes id))(x \otimes v) = \sum_{k=1}^{d_q} \Gamma^q(x)_{kj} Q_k^q(v) \quad \text{(B10)}$$

for all $x \in \mathcal{A}'$, all $v \in V$, and all $j = 1, 2, \ldots, d_q$. (Here π' is the mapping of $V \otimes \mathcal{A}'$ into V that coincides with π'^p on V^p and with π'^r on V^r, and which acts similarly on any other carrier spaces that might be contained in V). The generalization of (31) to this situation is obviously

$$((id \otimes M) \circ (\pi \otimes id) \circ (Q_j^q \otimes S) \circ \pi)(v) = \sum_{k=1}^{d_q} Q_k^q(v) \otimes \pi_{kj}^q \quad \text{(B11)}$$

for all $v \in V$ and all $j = 1, 2, \ldots, d_q$.

Because M is indistinguishable from $M \circ \sigma$ and S is indistinguishable from S^{-1} in the situation being considered here, the above arguments would equally well apply with each of the following 3 substitutions:

1. replace M by $M \circ \sigma$, but leave S unchanged;

2. leave M unchanged, but replace S by S^{-1};

3. replace M by $M \circ \sigma$ *and* replace S by S^{-1}.

However, in the general case in which $\mathcal{A}$ is non- commutative, the possibilities (1) and (2) are *excluded* because with them, and in the situation discussed in the previous paragraph, the identity operator would not be an irreducible tensor operator belonging to the identity corepresentation. With the substitution (3), (31) changes into (36), which is the defining condition for a *twisted* irreducible tensor operator $\widetilde{Q}_j^q$. (Of course the corresponding substitutions for $\mathcal{A}'$ are $\Delta_{\mathcal{A}'} \to \sigma \circ \Delta_{\mathcal{A}'}$ and $S_{\mathcal{A}'} \to (S'_{\mathcal{A}})^{-1}$, so that the analogues of (B4) and (B10) are

$$(\pi'^{r} \circ (id \otimes \widetilde{Q}^q_j) \circ (id \otimes \pi'^{p}) \circ (id \otimes (S'_{\mathcal{A}})^{-1} \otimes id) \circ (\sigma \otimes id) \circ (\Delta_{\mathcal{A}'} \otimes id))(x \otimes v^p) \\ = \textstyle\sum_{k=1}^{d_q} \Gamma^q(x)_{kj} \widetilde{Q}^q_k(v^p) \tag{B12}$$

(for all $x \in \mathcal{A}'$, all $v^p \in V^p$, and all $j = 1, 2, \ldots, d_q$), and

$$(\pi' \circ (id \otimes \widetilde{Q}^q_j) \circ (id \otimes \pi') \circ (id \otimes (S'_{\mathcal{A}})^{-1} \otimes id) \circ (\sigma \otimes id) \circ (\Delta_{\mathcal{A}'} \otimes id))(x \otimes v) \\ = \textstyle\sum_{k=1}^{d_q} \Gamma^q(x)_{kj} \widetilde{Q}^q_k(v) \tag{B13}$$

(for all $x \in \mathcal{A}'$, all $v \in V$, and all $j = 1, 2, \ldots, d_q$).

It is worth noting that (B4), (B10), (**??**), and (B13) provide the appropriate definitions for irreducible tensor operators not merely for the context in which they have been derived here (i.e. for the case in which $\mathcal{A}'$ is the group algebra of a finite group $\mathcal{G}$), but also for the case in which $\mathcal{A}'$ is the universal enveloping algebra $U(\mathcal{L})$ of a Lie algebra $\mathcal{L}$ (with $S_{\mathcal{A}'}(a) = -a$ and $\Delta_{\mathcal{A}'}(a) = a \otimes 1 + 1 \otimes a$ for all $a \in \mathcal{L}$), and for deformations of such universal enveloping algebras. (Of course in the case $\mathcal{A}' = \mathcal{U}(\mathcal{L})$, the criteria (B4) and (**??**) coincide and the criteria (B10) and (B13) also coincide, but this will not be true for deformations of $U(\mathcal{L})$).

APPENDIX C: DERIVATION OF THE RIGHT COACTIONS

Consideration of (3) suggests that one first defines an operator $\pi'_{\mathcal{L}^{pr}}$ to be the mapping of $\mathcal{A}' \otimes \mathcal{L}^{pr}$ into $\mathcal{L}^{pr}$ that is given by

$$\pi'_{\mathcal{L}^{pr}}(x \otimes Q) = \hat{\pi}'^{r}(x)\; Q\; \hat{\pi}'^{p}(x^{-1})\ , \tag{C1}$$

where the operators $\hat{\pi}'^{p}(x)$ were defined (B2) (and the $\hat{\pi}'^{r}(x)$ are defined similarly), and where Q is any member of $\mathcal{L}^{pr}$. If $a^1, a^2, \ldots$ form a basis for $\mathcal{A}'$, (C1) can be re-expressed in purely Hopf algebra terms (with $x = a^k$) as

$$\pi'_{\mathcal{L}^{pr}}(a^k \otimes Q) = \\ (\widehat{M} \circ (id \otimes \widehat{M}) \circ (\hat{\pi}'^{r} \otimes id \otimes \hat{\pi}'^{p}) \circ (id \otimes \sigma) \circ (id \otimes S_{\mathcal{A}'} \otimes id) \circ (\Delta_{\mathcal{A}'} \otimes id))(a^k \otimes Q). \tag{C2}$$

(Here the operator multiplication operation $\widehat{M}$ is defined by $\widehat{M}(Q \otimes Q') = Q \circ Q'$ for all $Q, Q' \in \mathcal{L}^{pr}$). It is then easily demonstrated that $\pi'_{\mathcal{L}^{pr}}$ is a *left action* of $\mathcal{A}'$ with carrier space $\mathcal{L}^{pr}$.

After some algebra, it can be shown that (C2) can be rewritten in terms of components as

$$\pi'_{\mathcal{L}^{pr}}(a^k \otimes Q) = \sum_{i,j=1}^{d_p} \sum_{m,n=1}^{d_r} \langle a^k \, , \, M(\pi^r_{mn} \otimes S(\pi^p_{ij})) \rangle \; q_{ni} \mathcal{P}^{pr}_{jm} \; , \tag{C3}$$

where the operators $\mathcal{P}^{pr}_{bi}$ are defined in (14) and the matrix elements q_{ja} are defined in (17). The corresponding right coaction $\pi_{\mathcal{L}^{pr}}$ of $\mathcal{A}$ is then given (c.f. (I.A6)) by

$$\pi_{\mathcal{L}^{pr}}(Q) = \sum_k \pi'_{\mathcal{L}^{pr}}(a^k \otimes Q) \otimes a_k \tag{C4}$$

for all $Q \in \mathcal{L}^{pr}$, where $a_1, a_2, \ldots$ is the dual basis of $\mathcal{A}$. Thus, by (C3) and (C4),

$$\pi_{\mathcal{L}^{pr}}(Q) = \sum_{i,j=1}^{d_p} \sum_{m,n=1}^{d_r} (q_{ni} \, \mathcal{P}^{pr}_{jm}) \otimes M(\pi^r_{mn} \otimes S(\pi^p_{ij})) \; ,$$

which is (32).

On replacing M by $M \circ \sigma$ and S by S^{-1}, the definition (32) changes into the definition (37) for $\widetilde{\pi}_{\mathcal{L}^{pr}}$.

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